What can cryptography do for coding theory?

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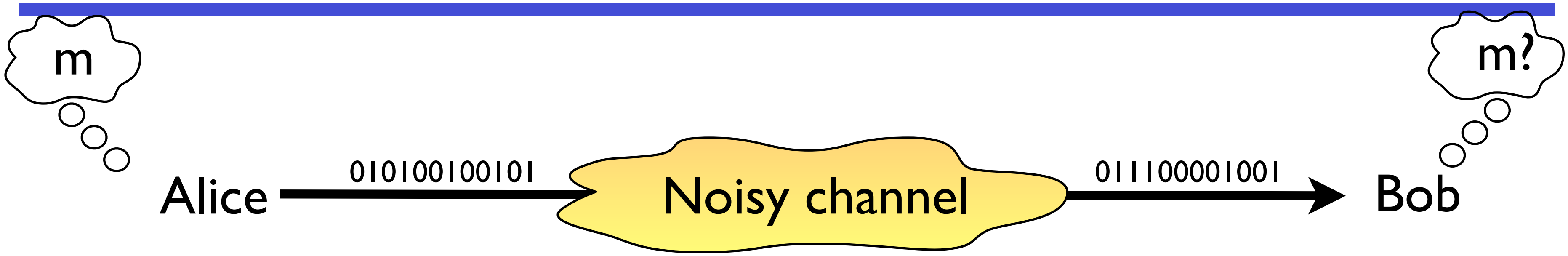
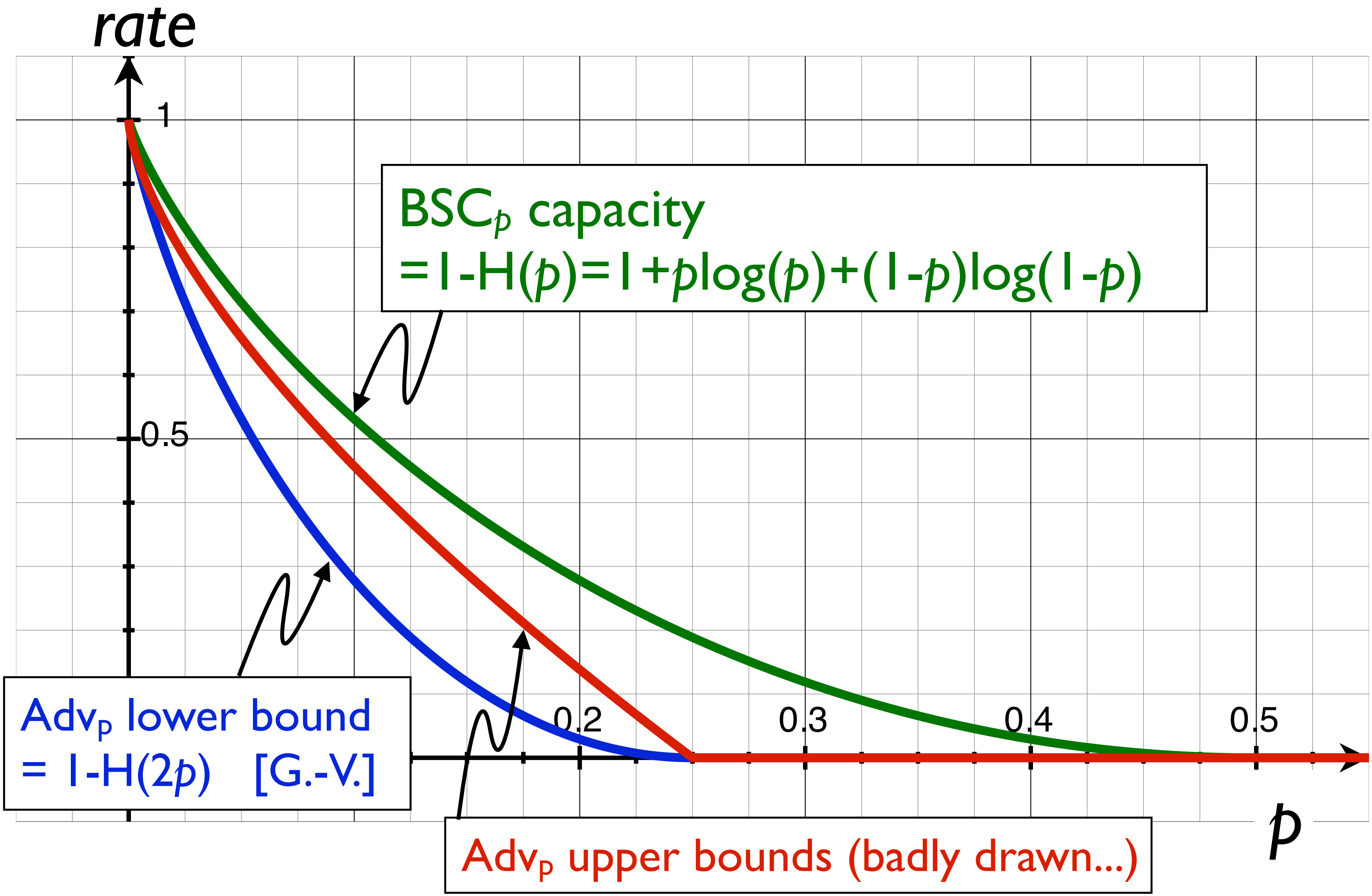
ICITS 2009

Two classic channel models



- Alice sends n bits
- Binary symmetric channel BSC_p
  - Flips each bit with probability p
  - Shannon: maximum possible rate is 1-H(p)
  - Forney: concatenated codes achieve capacity efficiently
- Worst-case (adversarial) errors ADV_p
  - Channel outputs an arbitrary word within distance pn of input
  - Optimal rate still unknown

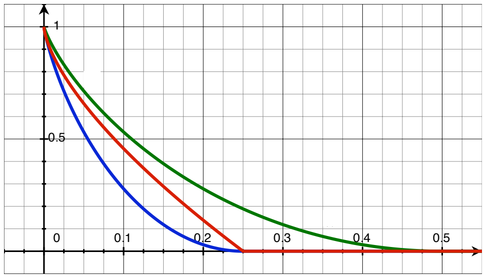
Known Bounds



Why care about worst-case errors?

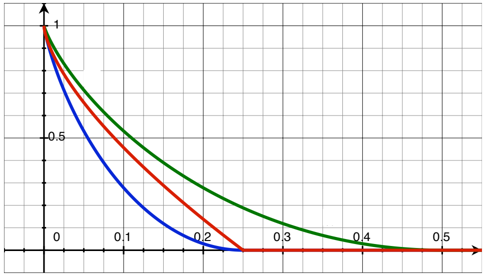
- Combinatorial interest
  - ➢ key building block for designs, authentication schemes, etc

- Modeling **unknown** or **varying** channels
  - ➢ Codes designed for one channel may fail if model wrong
  - ➢ *E.g.*, concatenated codes do badly against bursty errors



This talk: cryptographic tools in coding

- Models of uncertain binary channels
  - ➢ strong enough to capture wide variety of channel behavior
  - ➢ But reliable communication at Shannon capacity
- Theme: cryptographic perspective
  - ➢ modeling "limited" adversarial behavior
  - ➢ simpler existence proofs
  - ➢ techniques for efficient constructions: indistinguishability, pseudorandomness



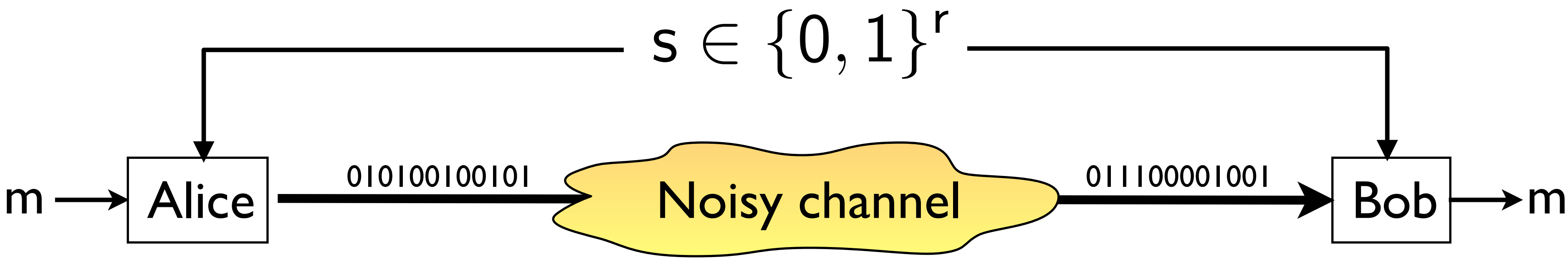
- Two kinds of models:
  - shared secrets for Alice and Bob
  - limited channels
- Two basic techniques
  - Sieving list decodable codes
  - “Scrambling” (randomizing) adversarial errors

Outline

- Developing tools: shared secrets
- Computationally limited channels
- Recent results:
  - ➢ Explicit constructions for worst-case "additive" errors [GS'09]
  - ➢ Efficient list-decoding for logspace channels [forthcoming]

Shared Randomness

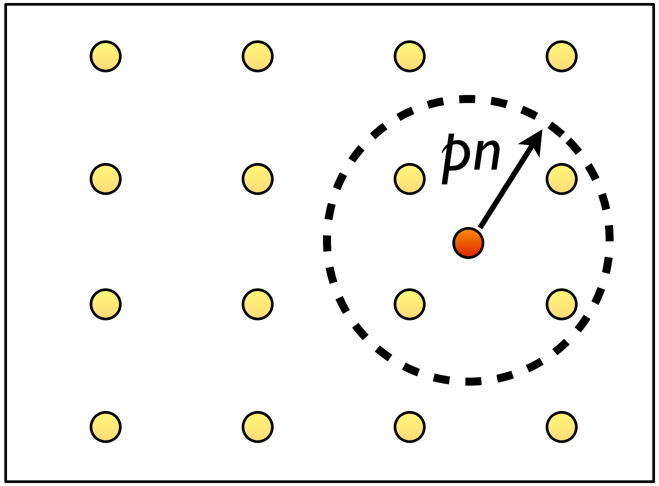
Shared Randomness



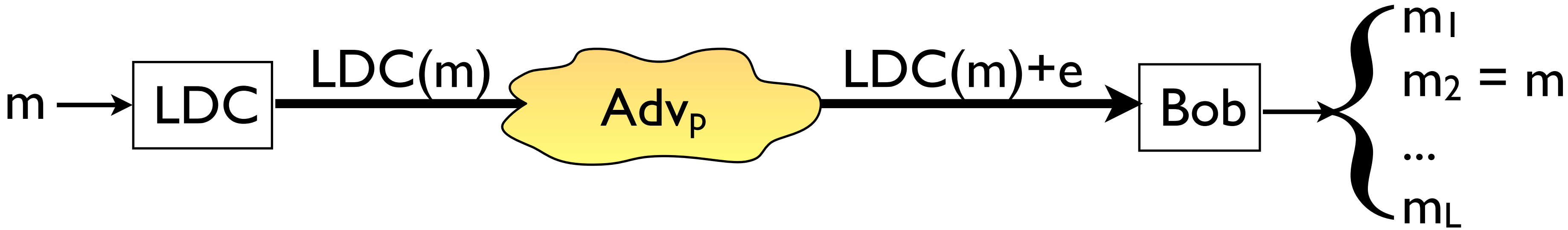
- Encoder/decoder share random bits s
  - code is known to channel but s is unknown
- **Theorem 1** [Langberg '04, ?]: With r=O(log n) shared bits, Alice can send ≈n(1-H(p)) bits reliably over Adv_p.
  - (not necessarily computationally efficient)
- A simple "cryptographic" proof
  - Tools: list-decoding, message authentication

Tool: List-decodable codes

- A code LDC: $\{0,1\}^k \rightarrow \{0,1\}^n$ is (pn,L) list-decodable code if
  - ➢ Every vector in $\{0,1\}^n$ is within distance pn of at most L codewords



- With LDC, Bob gets a list of L possible codewords



- **Proposition** [Elias]: There exist (pn,L) list-decodable codes with rate $1\text{-}H(p)\text{-}\varepsilon$ and list size $L = 1/\varepsilon$.

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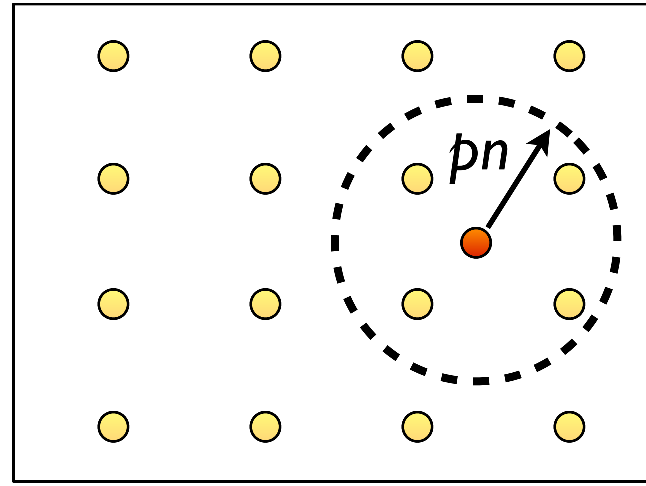

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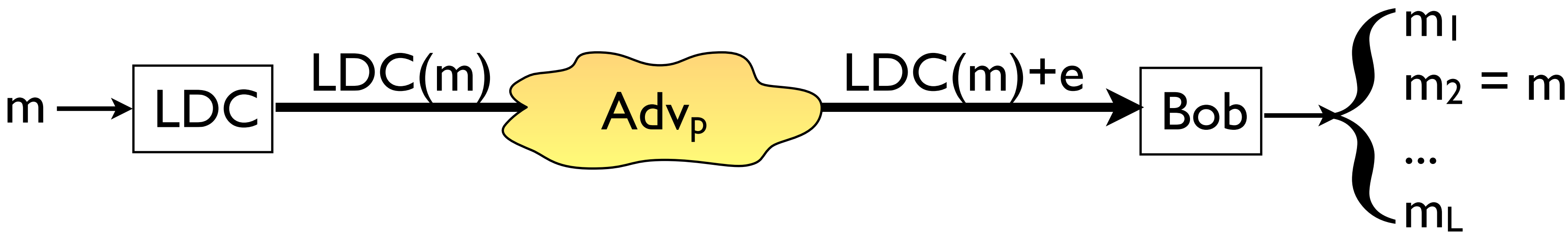
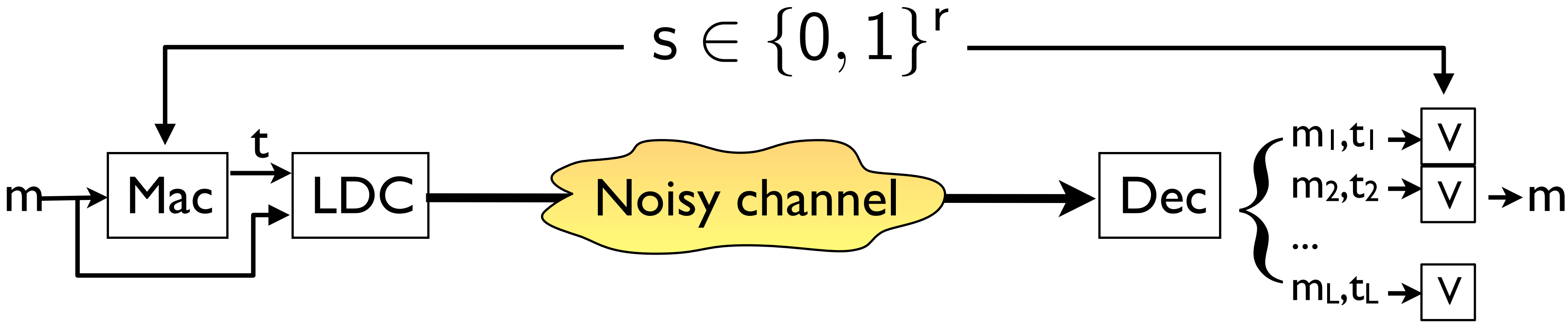


- **Proposition** [Elias]: There exist (pn,L) list-decodable codes with rate 1-H(p)-ε and list size L = 1/ε.

How can Bob figure out which is the right codeword?

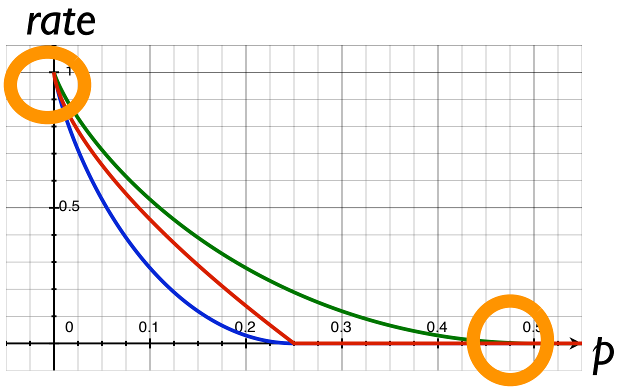
Sieving the List



- Idea: Alice authenticates m using s as key
- **Theorem 1** [Langberg '04, ?]: With r=O(log n) shared bits, Alice can send ≈n(1-H(p)) bits reliably over Adv_p.
- **Proof:** If MAC has forgery probability δ, then Bob corrects Adv_p errors with probability ≤ L δ
  - ➢ Adversary gets at most L chances to forge
  - ➢ MAC tag can have tag/key length O(log n)

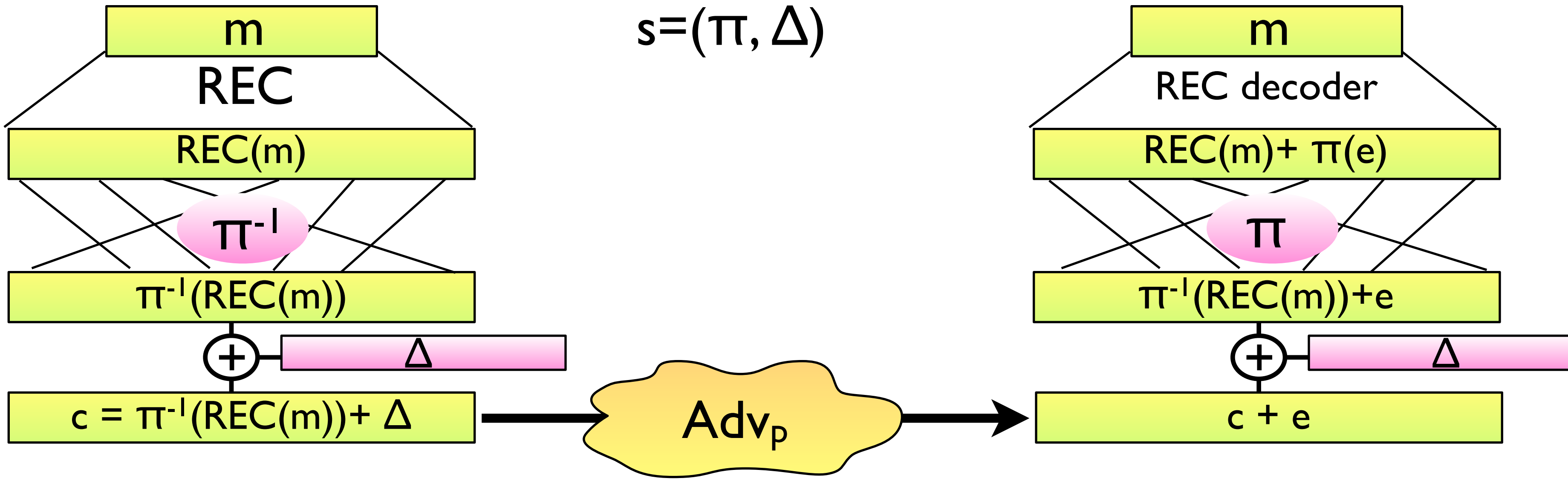
Computational Efficiency?

- Problem with list-decoding: **efficient** constructions only known for $p \approx 0$ and $p \approx 1/2$
  - for other values of p, efficient constructions have rate well below capacity



- **Theorem 2** [Lipton'94]: With $r \approx n \log(n)$ shared bits, Alice can Bob can efficiently and reliably communicate $\approx n(1\text{-}H(p))$ bits over Adv_p

Technique #2: Code Scrambling



- Shared randomness to permute errors randomly
  - ➢ Code REC corrects random errors with rate 1-H(p) [Forney]
  - ➢ $s=(\pi, \Delta)$ where π is a random permutation of $\{1,\ldots,n\}$

    and Δ is a random offset in $\{0,1\}^n$
  - ➢ Encoding: $c = \pi^{-1}(\mathrm{REC}(m)) + \Delta$

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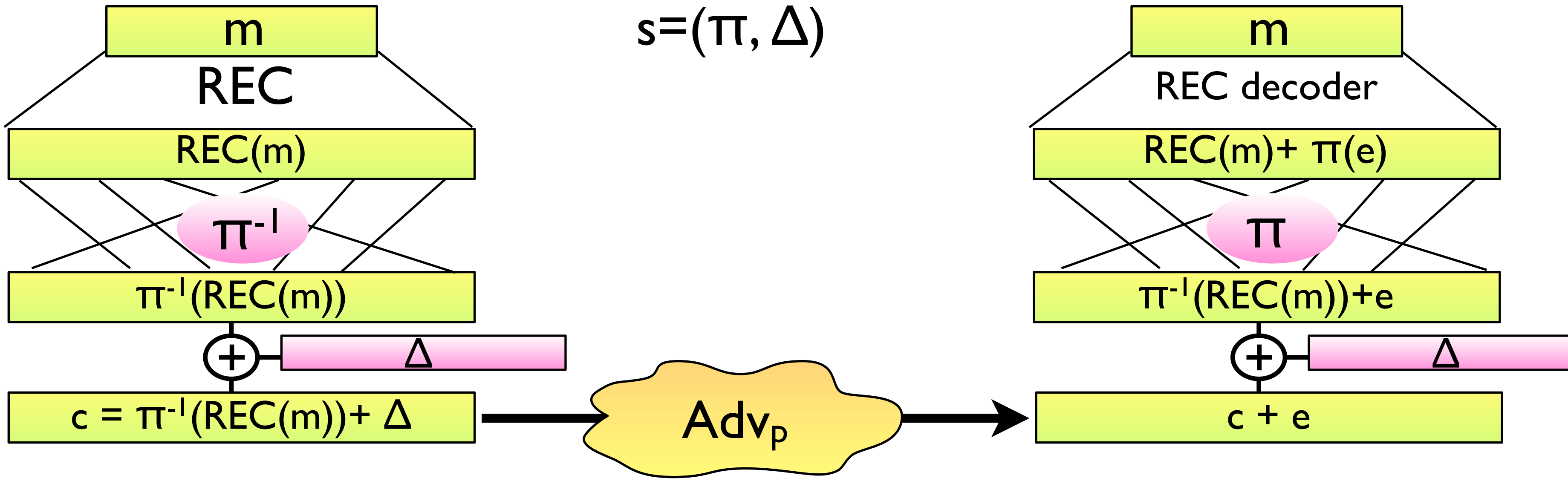
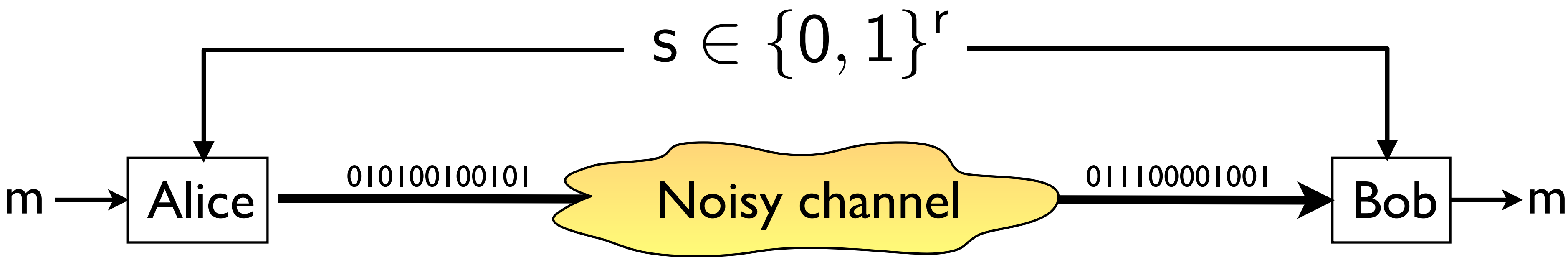


- **Theorem 2** [Lipton]: Scrambled code corrects pn adversarial errors with rate ≈ 1-H(p)
- **Proof**: Δ acts as one-time pad
  - ➢ e is independent of π
  - ➢ π(e) is a uniformly random vector of same weight as e (< pn)

Computational Efficiency w/ Short Keys?

- Code scrambling uses a long key: $\log(n!) + n$ bits
- **Open Question:** Can we get efficient codes of rate $n(1-H(p))$ that correct pn errors with keys of $o(n)$ bits?
- **Partial Answer**: $n+o(n)$ bits of key suffice
  - ➢ π just has to random enough to "fool" the REC decoder
  - ➢ **Lemma** [S'07]: Concatenated codes corrected $\log(n)$-wise independent errors up to Shannon capacity
  - ➢ π only has to be a $\log(n)$-wise independent permutation
  - ➢ **Lemma** [KNR'05]: $\log^2(n)$ bits suffice to select π
  - ➢ Get keys of length $n + \log^2(n)$... bottleneck is one-time pad!

Shared Randomness



- Can correct adversarial errors up to Shannon capacity
  - Two techniques: sieving list and code scrambling

Scheme	Key length	Efficient?
Sieving list	log(n)	No*
Scrambling	n log(n)	Yes
Scrambling with t-wise π	n + $\log^2$(n)	Yes

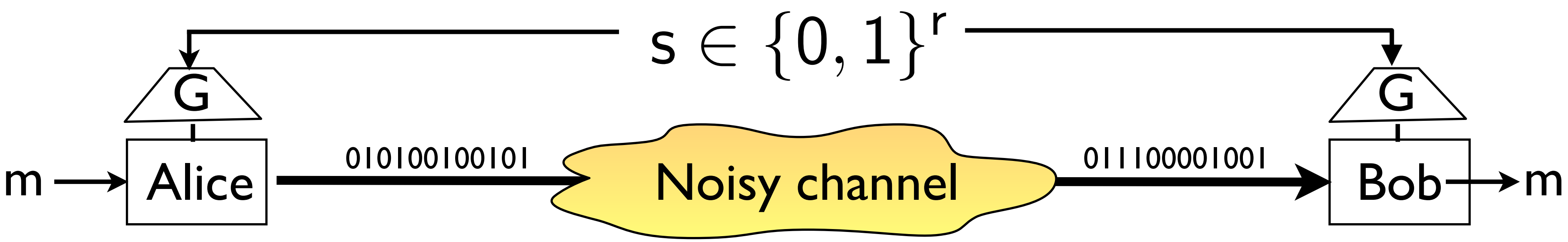
Limited Channels

Limited channels

- Idea: consider adversarial yet limited class of channels
  - processes in nature may vary in strange ways
    but they are **computationally simple**
- Polynomial-time channels [Lipton]
  - Can strengthen results for shared randomness model
  - Models with no setup?
- Additive channels [A,CN]
  - Model noise that is oblivious to individual bits
  - Explicit, poly-time constructions

Polynomial-time Adversaries

- Shared key setting [Lipton]
  - Use a p.r.g. to do code scrambling with a short seed
  - Get O(log n)-bit keys and efficient decoding (assuming OWF)



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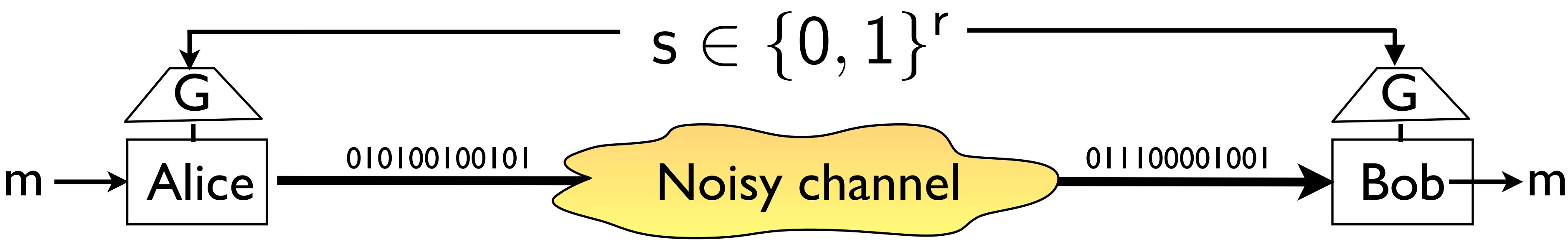
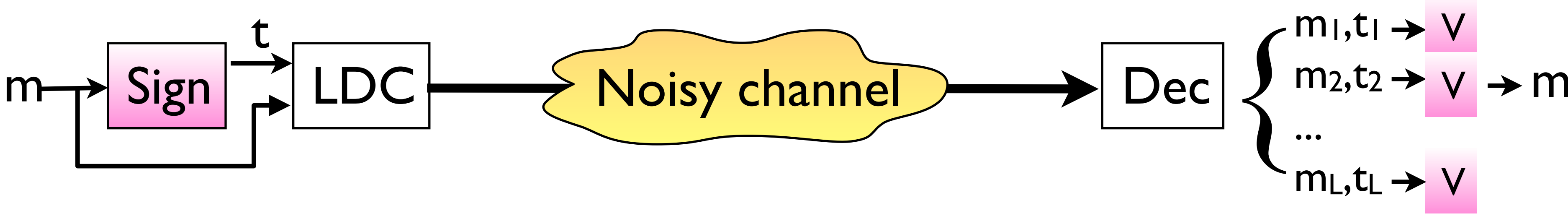


- Public key setting [Micali, Peikert, Sudan, Wilson]
  - ➢ Alice broadcasts a public key; keeps a secret key
  - ➢ Replace MAC with signatures in list-sieving

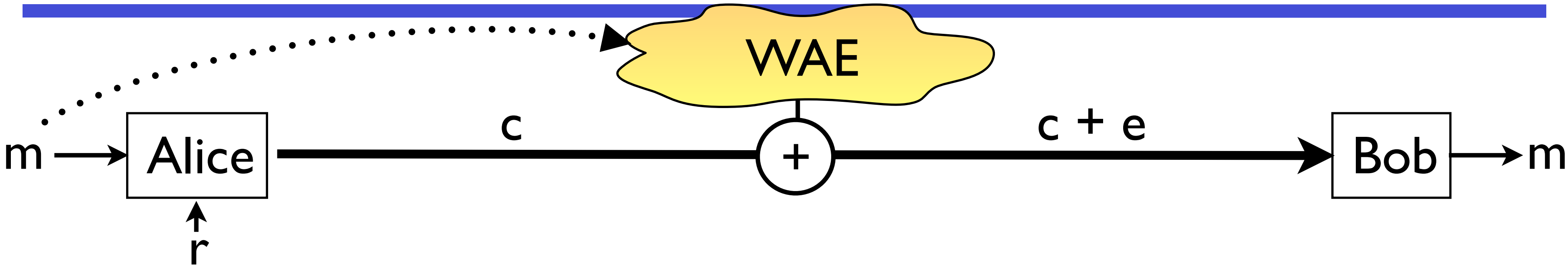


What about models without setup?

- Nothing (significant) is known regarding polynomial time
  - Some bounds clearly apply (e.g. Shannon bound)
  - Unclear if one can beat information-theoretic bounds for adversarial channels

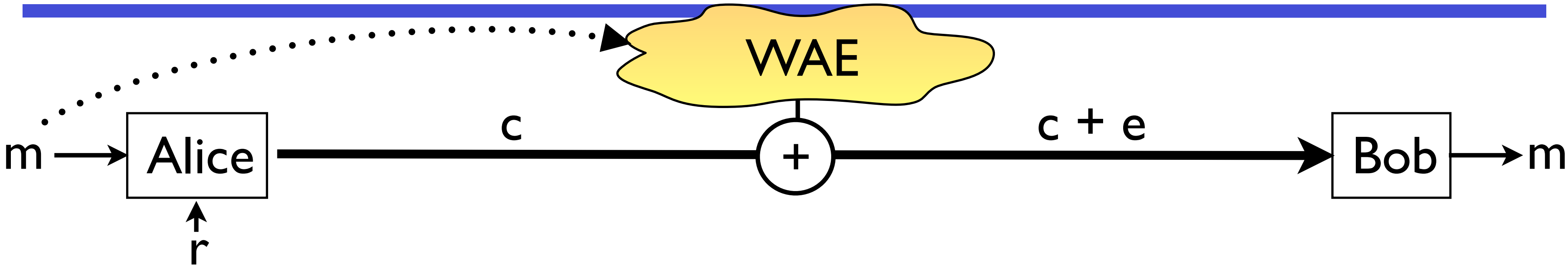
- Different extreme: **additive channels**
  - **very simple** channels
  - error pattern is adversarial, but independent of codeword

Worst-case additive errors



- Adversary picks error pattern e of weight < pn **before seeing codeword**
  - ➢ Adversary knows **code** and **message**
  - ➢ Alice generates local random bits ***r*** (unknown to Bob/channel)
- Generalizes natural symmetric error models
  - ➢ e.g., BSC, burst errors
- Natural step towards general classes of functions
- Special case of state-constrained AVC [Csiszár-Narayan]

Worst-case additive errors



- AVC's literature has general upper/lower bounds
- **Theorem** [Csiszár-Narayan, Langberg]: There exist codes with rate ≈1-H(p) that correct pn additive errors.
    - ➢ Complex random coding arguments
- [Guruswami-S., '09]: This talk
    - ➢ Simpler existence proof via sieving LDC's
    - ➢ Explicit construction with efficient encoding / decoding

Tool: Algebraic Manipulation Detection [CDFPW'08]

- "Error detection" for additive errors
- Randomized encoding AMD: $m \longmapsto$ AMD(m,r)
  - Verify(AMD(m,r)) = 1 always
  - For all fixed error patterns e, w.h.p. over r,

    Verify(AMD(m,r)+e)=false
- Simple construction expands m by O(log(n)) bits
  - Use m to choose coefficients of low-degree polynomial f_m
  - AMD(m,r) = (m, r, $f_m(r)$)
  - **Lemma** [DKRS]: If we ensure that the leading coefficients of f_m have the right form, then for all m and for all offsets a,b,c:

    Pr($f_{m+a}(r+b) = f_m(r)+c$) is small

Good codes for additive errors [GS'09]

- Use AMD scheme to sieve list of linear LDC

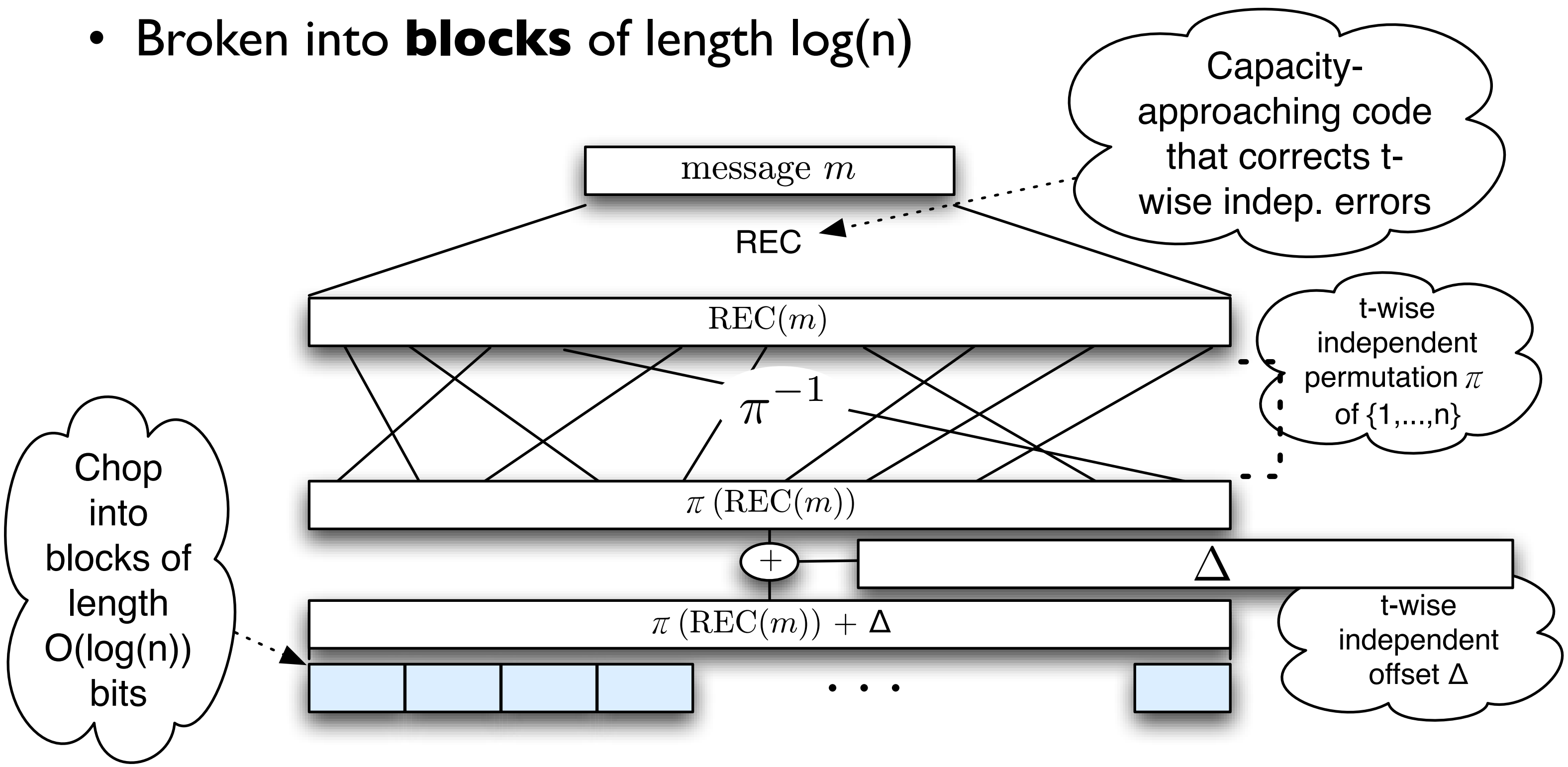
- This corrects as many errors as LDC
  - For any string x, Dec(LDC(x)+e) = $\{x, x+e_2, \ldots x+e_L\}$
  - Since LDC is linear, errors $e_2, \ldots, e_L$ independent of x
  - AMD rejects all non-zero errors w.h.p.
- **Lemma** [Guruswami-Hastad-Sudan-Zuckerman]: There exist linear LDC with rate 1-H(p)-ε and list size O(1/ε).
- Consequence: additive errors codes w. rate 1-H(p) exist

Efficient Constructions

- List-decoding construction not efficient in general
  - Would like to get to capacity for all error rates p
- Idea:
  - bootstrap from "small" code (decodable by brute force) to "big code" (decodable efficiently)
  - Standard tool: concatenation [Forney]
    - Use big code over large alphabet + small code to encode symbols
    - Concatenation works poorly for worst-case errors
    - Adversary can concentrate errors in blocks (e.g. bursts)
  - Instead: use small code to share secret key for scrambling
    - Interleave small code blocks into big code blocks pseudorandomly

Control/payload construction

- Two main pieces
  - Scrambled "payload codeword": $\pi^{-1}(\text{REC}(m)) + \Delta$
    - π is a $\log^2(n)$-wise independent permutation,
    - Δ is a $\log^2(n)$-wise independent bit string
    - Broken into **blocks** of length log(n)



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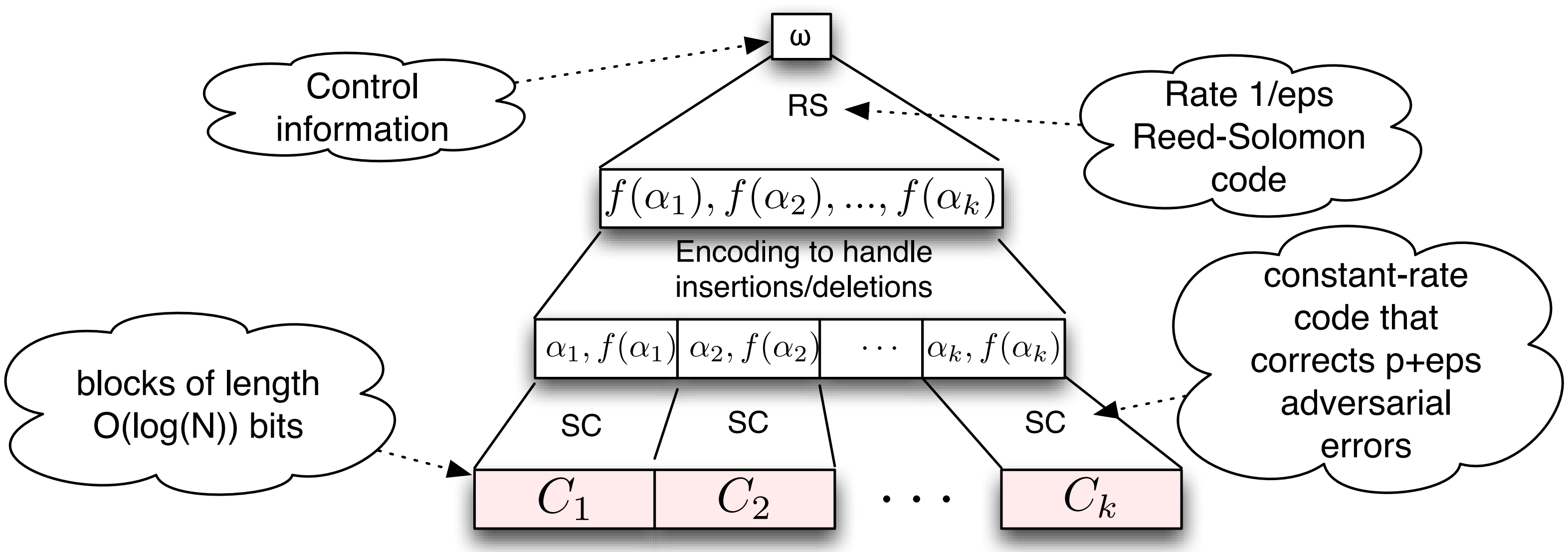
  - ➢ Scrambled "payload codeword": $\pi^{-1}(REC(m)) + \Delta$
  - ➢ "Control information": $\omega = (\pi, \Delta, T)$
    - T is a set of blocks in $\{1,\ldots, n/\log(n)\}$
    - ω is encoded using Reed-Solomon-code into "control blocks"
    - Each control block encoded using small LDC+AMD code



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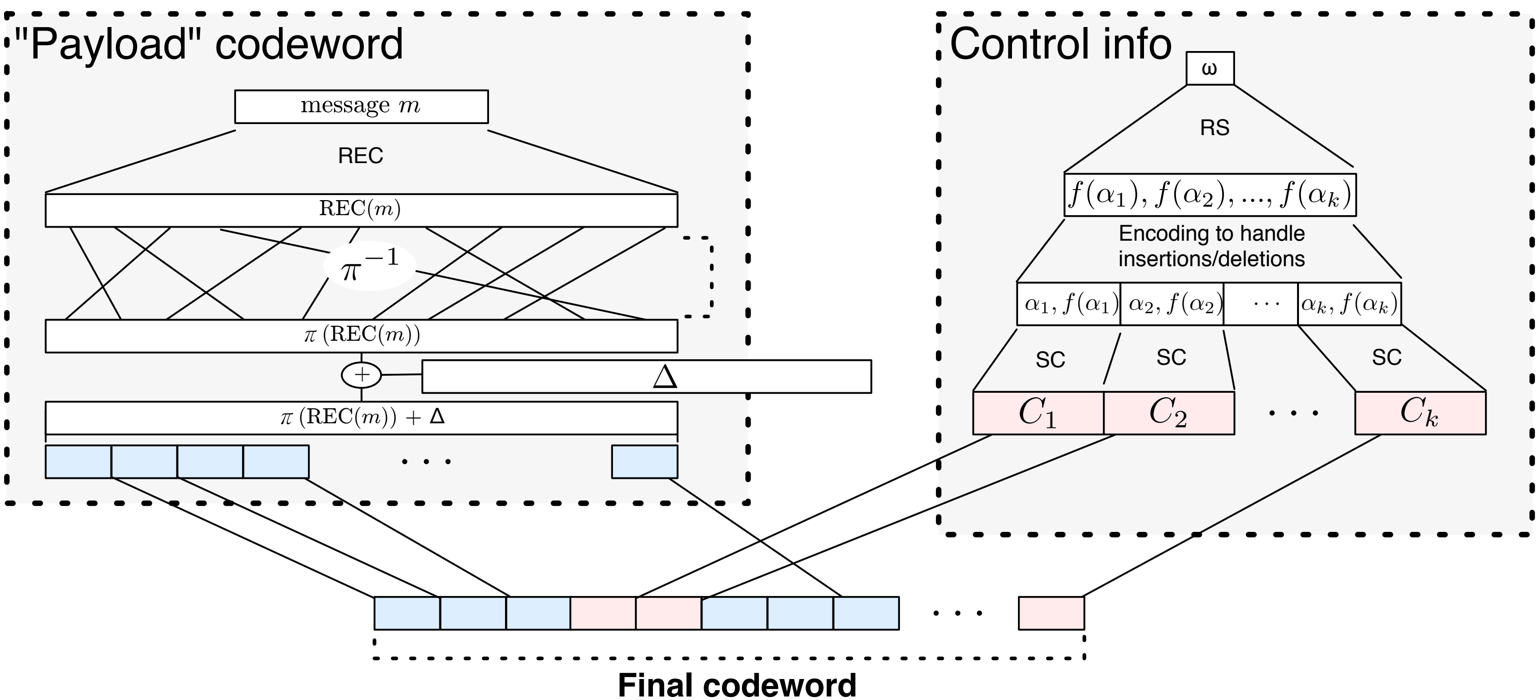
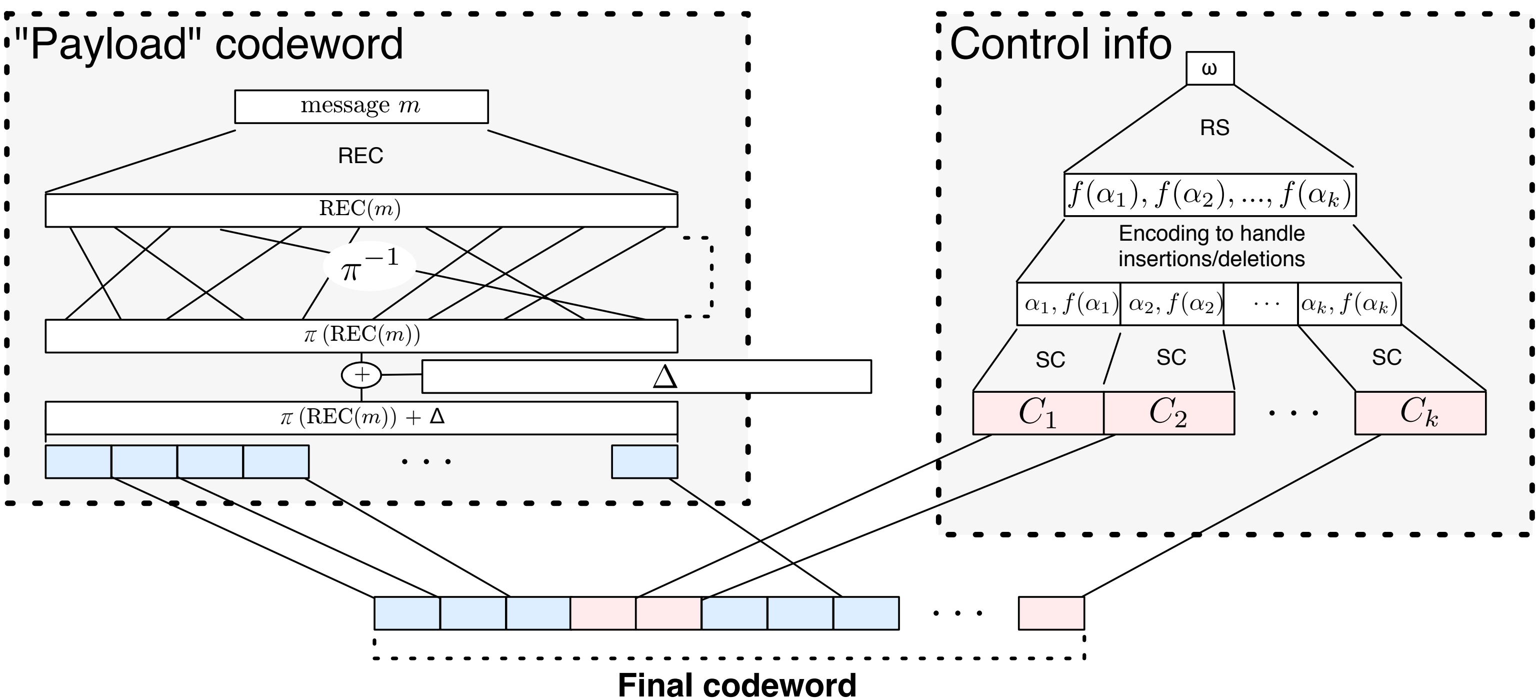
- Combine by interleaving according to T



Control/payload Construction

- Decoding idea
  - ➢ First decode control information, block by block
  - ➢ Given control information, unpermute scrambled code



Outline

- Developing tools: shared secrets
- Computationally limited channels
- Recent results:
  - ➢ Explicit constructions for worst-case “additive” errors [GS’09]
  - ➢ Efficient list-decoding for logspace channels [forthcoming]

Logspace channels

- Additive channels natural but maybe too limited
  - ➢ What if channel **sets** bits to 0/1?
  - ➢ Flips 0 to 1 more often than 1 to 0?
- Limited-memory channels
  - ➢ Errors introduced online, as codeword passes through channel
  - ➢ Channel can only remember t bits
  - ➢ Modeled as branching program with width 2^t
  - ➢ t = O(log n) captures every channel I can think of...
  - ➢ t=n: "online channels" [Langberg], known to be quite powerful
- Can we achieve Shannon capacity?

Conclusions

- Models for achieving maximum transmission rates in binary channels, despite uncertain or adversarial channel behavior

- Perspective, tools from cryptography / derandomization
  - ➢ Disciplinary lines are artificial
  - ➢ Crypto / information theory communities share many questions and techniques
  - ➢ But also lots of ideas take time to cross over